Prolongation structure of the Krichever-Novikov equation

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ABSTRACT. We completely describe Wahlquist-Estabrook prolongation structures (coverings) dependent on u, u_x, u_{xx}, u_{xxx} for the Krichever-Novikov equation $u_t = u_{xxx} - 3u_{xx}^2/(2u_x) + p(u)/u_x + au_x$ in the case when the polynomial $p(u) = 4u^3 - g_2u - g_3$ has distinct roots. We prove that there is a universal prolongation algebra isomorphic to the direct sum of a commutative 2-dimensional algebra and a certain subalgebra of the tensor product of $\mathfrak{sl}_2(\mathbb{C})$ with the algebra of regular functions on an affine elliptic curve. This is achieved by identifying this prolongation algebra with the one for the anisotropic Landau-Lifshitz equation. Using these results, we find for the Krichever-Novikov equation a new zero-curvature representation, which is polynomial in the spectral parameter in contrast to the known elliptic ones.

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1. Introduction

The Krichever-Novikov (KN) equation

$$(1)\quad u_t = u_3 - \frac{3}{2}\frac{u_2^2}{u_1} + \frac{4u^3 - g_2u - g_3}{u_1} + au_1, \quad u_k = \frac{\partial^k u}{\partial x^k}, \quad g_2,\, g_3,\, a \in \mathbb{C},$$

appeared for the first time in [**6**] in connection with a study of finite-gap solutions of the KP equation. If the roots e_1, e_2, e_3 of the polynomial $4u^3 - g_2u - g_3$ are distinct then equation (1) is called *nonsingular*. According to [**11, 12**], in this case no differential substitution

$$(2)\qquad \tilde{u} = g(u, u_1, u_2, \dots)$$

exists connecting (1) with other equations of the form

$$(3)\qquad u_t = u_3 + f(u, u_1, u_2).$$

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Moreover, nonsingular equations (1) exhaust (up to invertible transformations $u = \varphi(\tilde{u})$) all the *integrable* (possessing an infinite series of conservation laws) equations (3) that are not reducible by a finite number of substitutions (2) to the KdV equation $u_t = u_3 + u_1 u$ or the linear equation $u_t = u_3 + au_1$.

These distinctive features make equation (1) worth to study in detail. In this paper we apply the Wahlquist-Estabrook prolongation method to it. Some particular zero-curvature representations [**4, 6, 7**] as well as a Bäcklund transformation [**1**] for (1) are known, but a complete description of prolongation structures has not been given, and we perform this below.

It turns out that with respect to prolongation structures equation (1) continues to demonstrate remarkable properties. First of all, in order to obtain nontrivial results one has to consider prolongation structures of order 3 (i.e., dependent on u_k, $k \leq 3$) in contrast to the normal assumption that their order is lower than the equation's order. Because of this, there is additional gauge freedom, which impedes the computation. Fortunately, we find a canonical form for the considered prolongation structures, which fixes partially the gauge and makes it possible to obtain a universal prolongation algebra in terms of generators and relations. In Section 3 we show that this is in fact the case for any equation of the form $u_t = u_3 - 3u_2^2/(2u_1) + f(u)/u_1 + au_1$.

Following [**5, 2**], in order to clarify the computation and the nature of gauge transformations we interpret differential equations as submanifolds in infinite jet spaces and prolongation structures as special morphisms called *coverings* of such manifolds. This method is recalled in Section 2.

In [**4, 6**] it is noticed that Sklyanin's zero-curvature representation for the anisotropic Landau-Lifshitz (LL) equation leads by means of a special transformation of the dependent variables to a zero-curvature representation for the nonsingular equation (1). Note that this is not a Bäcklund transformation and does not establish any correspondence between solutions of the two equations.

In Section 5 we make use of this transformation to choose special generators in the prolongation algebra $\mathfrak{g}$ of (1) in the nonsingular case such that the resulting relations turn into the ones for the LL prolongation algebra. In [**9**] the latter algebra was explicitly described, and we recall this description in Section 4.

Finally, in Section 5 we prove that $\mathfrak{g}$ is isomorphic to the direct sum of a commutative 2-dimensional algebra and a certain subalgebra of the tensor product of $\mathfrak{sl}_2(\mathbb{C})$ with the ring $\mathbb{C}[v_1, v_2, v_3]/\mathcal{I}$, where the ideal $\mathcal{I}$ is generated by the polynomials

$$v_i^2 - v_j^2 + \frac{8}{3}(e_j - e_i), \quad i,\, j = 1, 2, 3,$$

defining a nonsingular elliptic curve in $\mathbb{C}^3$.

In particular, we establish one-to-one correspondence between zero-curvature representations (ZCR) for the anisotropic LL equation and the nonsingular KN equation. Using this, in Section 6 we derive a new $\mathfrak{sl}_4(\mathbb{C})$-valued ZCR for the nonsingular KN equation from the found in [**3**] ZCR for the LL equation. Remarkably, this ZCR is polynomial in the spectral parameter in contrast to the known for (1) ZCR with elliptic parameters [**4, 6, 7**].

Generally, we think that, side by side with the symmetry algebra and the space of conservation laws, the prolongation algebra is an important invariant of a given system of differential equations.

The obtained algebra $\mathfrak{g}$ differs considerably from other known prolongation algebras for equations of the form (3). Indeed, for the KdV equation and the potential KdV equation $u_t = u_3 + u_1^2$ the Lie algebras governing the prolongation structures of order 2 were described explicitly in [**13**] and [**8**] respectively. In both cases the algebra turned out to be the direct sum of the polynomial loop algebra $\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathbb{C}[\lambda]$ and a finite-dimensional nilpotent algebra.

2. Prolongation structures as coverings

We use the following modification due to Krasilshchik and Vinogradov [**5, 2**] of the original Wahlquist-Estabrook method. Let $\mathcal{E} \subset J^\infty(\pi)$ be the (infinite-dimensional) submanifold determined by a system of differential equations and its differential consequences in the infinite jet space $J^\infty(\pi)$ of some smooth bundle $\pi\colon E \to U$, where U is an open subset of $\mathbb{R}^n$. Let $x_1, \dots, x_n$ be coordinates in U, which play the role of independent variables in the equations. The total derivative operators D_{x_i} are treated as commuting vector fields on $\mathcal{E}$.

A *covering over* $\mathcal{E}$ is given by a smooth bundle

$$\psi\colon \tilde{\mathcal{E}} \to \mathcal{E} \tag{4}$$

and an n-tuple of vector fields $\tilde{D}_{x_i}$, $i = 1, \dots, n$, on the manifold $\tilde{\mathcal{E}}$ such that

$$\psi_*(\tilde{D}_{x_i}) = D_{x_i}, \tag{5}$$

$$[\tilde{D}_{x_i}, \tilde{D}_{x_j}] = 0, \quad \forall\, i, j = 1, \dots, n. \tag{6}$$

See [**5**] for a motivation of this definition and its coordinate-free formulation.

A diffeomorphism $\varphi\colon \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ such that $\psi \circ \varphi = \psi$ is called a *gauge transformation*, and the covering given by $\{\varphi_*(\tilde{D}_{x_i})\}$ is said to be *gauge equivalent* to the covering $\{\tilde{D}_{x_i}\}$.

In this paper we consider equations in two independent variables x and t, i.e., $n = 2$. Then Wahlquist-Estabrook prolongation structures [**14**] correspond to the case when (4) is a trivial bundle

$$\psi_{\mathrm{tr}}\colon \mathcal{E} \times W \to \mathcal{E}, \quad \dim W = m < \infty.$$

Local coordinates $w^1, \dots, w^m$ in W correspond to *pseudopotentials* in the Wahlquist-Estabrook approach [**14**]. From (5) we have

$$\tilde{D}_x = D_x + A, \quad \tilde{D}_t = D_t + B, \tag{7}$$

where

$$A = \sum_j A^j \partial_{w^j}, \quad B = \sum_j B^j \partial_{w^j} \tag{8}$$

are ψ_{tr}-vertical vector fields. Condition (6) is written as

$$D_x B - D_t A + [A, B] = 0. \tag{9}$$

A covering gauge equivalent to the one given by $\tilde{D}_x = D_x$, $\tilde{D}_t = D_t$ is called *trivial*.

We call a ψ_{tr}-vertical vector field A on $\mathcal{E} \times W$ *linear* (with respect to the given system of coordinates in W) if $A = \sum_{ij} a_{ij} w^j \partial_{w^i}$ for some functions $a_{ij} \in C^\infty(\mathcal{E})$. Denote by A_M the $m \times m$ matrix-function on $\mathcal{E}$ with the entries $[A_M]_{ij} = a_{ij}$. For two linear vector fields A, B the commutator $[A, B]$ is also linear, and one has

$$[A, B]_M = [B_M, A_M]. \tag{10}$$

If the linear vector fields A, B meet (9) then the matrices A_M, B_M satisfy

$$[D_x - A_M,\, D_t - B_M] = D_t A_M - D_x B_M + [A_M, B_M] = 0. \tag{11}$$

and form a *zero-curvature representation* (ZCR) of $\mathcal{E}$. The functions A_M, B_M may in fact take values in an arbitrary Lie algebra $\mathfrak{g}$, and then the ZCR is said to be $\mathfrak{g}$-*valued*.

3. Coverings of KN type equations

In this section we solve (9) for equations of the form

$$u_t = u_3 - \frac{3}{2}\frac{u_2^2}{u_1} + \frac{p(u)}{u_1} + a u_1, \quad u_k = \partial^k u / \partial x^k, \tag{12}$$

where u is a complex-valued function of two real variables x, t and $p(u)$ is an arbitrary analytic function of u. In this case $\pi\colon \mathbb{C} \times \mathbb{R}^2 \to \mathbb{R}^2,\ (u, x, t) \mapsto (x, t)$.

Remark 1. Here the bundle π and its jet bundles are complex, while [**5, 2**] deal with real bundles. However, it is easily seen that the theory of coverings is the same for complex bundles. In our case this follows from the concrete formulas presented below.

The manifold $\mathcal{E}$ has the natural coordinates x, t, u_k, $k \geq 0$, where x, t are real and u_k are complex. The total derivative operators are

written in these coordinates as follows

$$D_x = \partial_x + \sum_{j\geq 0} u_{j+1}\partial_{u_j}, \tag{13}$$

$$D_t = \partial_t + \sum_{j\geq 0} D_x^j(F)\partial_{u_j}, \tag{14}$$

where F is the right-hand side of (12).

Below $w^1, \dots, w^m$ are also complex, and all functions and vector fields are complex-valued and analytic with respect to their complex arguments.

Studying coverings over an evolution equation $u_t = f(u, u_1, \dots, u_p)$, one normally assumes to simplify the problem that A, B in (9) do not depend on the variables x, t and the derivatives u_k, $k \geq p$. However, in order to obtain nontrivial coverings for equation (12) we have to allow A, B to depend at least on u_k, $k \leq 3$ (and, of course, on $w^1, \dots, w^m$), see Remark 4 below.

Then a straightforward computation shows that (9) requires $A = A(w, u, u_1, u_2, u_3)$ to be of the form

$$A = \frac{1}{u_1}A_1(w,u) + A_0(w,u) + u_1 A_2(w,u) \tag{15}$$

Here and in what follows the symbol w stands for the whole collection $w^1, \dots, w^m$. We want to get rid of the term $u_1 A_2(w, u)$ by switching to a gauge equivalent covering.

To this end, let $A_2(w,u) = \sum_j a^j(w^1, \dots, w^m, u)\partial_{w^j}$ and fix $u' \in \mathbb{C}$. Consider a local analytic solution of the system of ordinary differential equations

$$\frac{d}{du} f^j(w,u) = a^j(f^1, \dots, f^m, u), \quad j = 1, \dots, m, \tag{16}$$

dependent on the parameters w with the initial condition $f^j(w, u') = w^j$.

Then the formulas

$$u_k \mapsto u_k, \quad w^j \mapsto f^j(w,u), \quad k \geq 0,\ j = 1, \dots, m, \tag{17}$$

define a local gauge transformation $\varphi\colon \mathcal{E} \times W \to \mathcal{E} \times W$ such that $\varphi_*(D_x + A) = D_x + A'$, where the vector field A' is of the form (15) without the linear in u_1 term.

Remark 2. This is easily seen from the following interpretation of coverings [**5, 2**]. The manifold $\mathcal{E} \times W$ is itself isomorphic to the submanifold in an infinite jet space determined by the system consisting of equation (12) and the following additional equations

$$\begin{aligned} \frac{\partial w^j}{\partial x} &= A^j(w, u, u_1, u_2, u_3), \\ \frac{\partial w^j}{\partial t} &= B^j(w, u, u_1, u_2, u_3), \end{aligned} \qquad j = 1, \dots, m, \tag{18}$$

where A^j, B^j are the components of A, B in (8). The vector fields $D_x + A$, $D_t + B$ are the restrictions of the total derivative operators to $\mathcal{E} \times W$. Gauge transformations correspond to invertible changes of variables

$$w^j \mapsto g^j(x, t, w, u, u_1, \dots), \quad j = 1, \dots, m,$$

in (18). Clearly, due to equation (16) after substitution (17) the linear in u_1 terms contract in (18).

Since we are interested in local classification of coverings up to gauge equivalence, we can from the beginning assume that

$$A = \frac{1}{u_1} A_1(w, u) + A_0(w, u). \tag{19}$$

Remark 3. This rather unusual step in solving (9) is due to the assumption that A, B may depend on the derivatives u_k, $k \le 3$.

Further computation shows that $A_0(w, u)$ does not actually depend on u. Denote for brevity $A_1 = A_1(w, u)$ and $A_0 = A_0(w)$. Finally, we obtain

$$\begin{aligned} B = &-\frac{u_3}{u_1^2} A_1 + \frac{u_2^2}{2u_1^3} A_1 + \frac{2u_2}{u_1} \frac{\partial A_1}{\partial u} + \frac{u_2}{u_1^2} [A_0, A_1] \\ &- \frac{p(u)}{3u_1^3} A_1 + \frac{2}{u_1} [A_1, \frac{\partial A_1}{\partial u}] - 2u_1 \frac{\partial^2 A_1}{\partial u^2} + aA + B_0(w), \end{aligned} \tag{20}$$

where

$$\frac{\partial^3 A_1}{\partial u^3} = 0, \tag{21}$$

$$[A_0, B_0] = [A_1, A_0] = [A_1, B_0] = 0, \tag{22}$$

$$2p \frac{\partial A_1}{\partial u} - \frac{\partial p}{\partial u} A_1 - 3[A_1, [A_1, \frac{\partial A_1}{\partial u}]] = 0. \tag{23}$$

Remark 4. From (20) we see that if B does not depend on u_3 (i.e., $A_1 = 0$) then A, B do not depend on u_k at all and, therefore, the covering is trivial. Hence our assumption that A, B may depend on u_k, $k \le 3$, is essential.

From (21) we see that A_1 is a polynomial in u of degree not greater than 2, i.e.,

$$A_1 = A_{10} + uA_{11} + u^2 A_{12} \tag{24}$$

for some vector fields $A_{1j} = A_{1j}(w)$.

Thus any covering of the considered type is uniquely determined by 5 independent of u_k, $k \ge 0$, vector fields on W

$$A_0,\ B_0,\ A_{1j},\ j = 0, 1, 2, \tag{25}$$

subject to restrictions (22), (23). For each concrete function $p(u)$ equations (22) and (23) give some relations between vector fields (25). As

usual, the quotient of the free Lie algebra generated by letters (25) over these relations is called the *prolongation algebra* of equation (12). From (22) we see that A_0, B_0 lie in the center of the prolongation algebra.

In Section 5 we solve these relations in the case when $p(u)$ is a polynomial of degree 3 with distinct roots. To achieve this, the description of the prolongation algebra for the Landau-Lifshitz equation is needed, which we recall in the next section.

4. Prolongation structure of the Landau-Lifshitz equation

The Landau-Lifshitz (LL) equation reads [**4, 9**]

$$\mathbf{S}_t = \mathbf{S}\times\mathbf{S}_{xx} + \mathbf{S}\times J\mathbf{S}, \quad S_1^2+S_2^2+S_3^2=1, \tag{26}$$

where $\mathbf{S}=(S_1,S_2,S_3)$ is a complex-valued vector-function of x,t and $J=\operatorname{diag}(j_1,j_2,j_3)$, $j_k\in\mathbb{C}$, is a diagonal matrix.

For (26) equation (9) under the normal assumption that A, B do not depend on x, t and derivatives of $\mathbf{S}$ of order >1 was solved in [**9**] as follows

$$A = \mathbf{P}\cdot\mathbf{S} + P_4, \tag{27}$$

$$B = (\mathbf{P}\times\mathbf{S})\cdot\mathbf{S}_x + (\mathbf{P}\times\mathbf{P})\cdot\mathbf{S} + P_5, \tag{28}$$

where $\mathbf{P}=(P_1,P_2,P_3)$ and $\mathbf{P}\times\mathbf{P}=([P_2,P_3],[P_3,P_1],[P_1,P_2])$. Here the vector fields P_i have to satisfy the relations

$$[P_j,P_4]=[P_j,P_5]=0, \quad j=1,\dots,5, \tag{29}$$

and

$$\begin{aligned}
&[P_1,[P_2,P_3]]=[P_2,[P_3,P_1]]=[P_3,[P_1,P_2]]=0,\\
&[P_2,[P_2,P_3]]-[P_1,[P_1,P_3]]+(j_1-j_2)P_3=0,\\
&[P_3,[P_3,P_1]]-[P_2,[P_2,P_1]]+(j_2-j_3)P_1=0,\\
&[P_1,[P_1,P_2]]-[P_3,[P_3,P_2]]+(j_3-j_1)P_2=0.
\end{aligned} \tag{30}$$

In the *full anisotropy* case

$$j_1\neq j_2,\ j_2\neq j_3,\ j_3\neq j_1, \tag{31}$$

the Lie algebra defined by the generators P_1, P_2, P_3 and relations (30) was described in [**9**] explicitly as follows. Consider the ideal $\mathcal{I}\subset\mathbb{C}[v_1,v_2,v_3]$ generated by the polynomials

$$v_\alpha^2 - v_\beta^2 + j_\alpha - j_\beta, \quad \alpha\,,\beta=1,2,3, \tag{32}$$

and set $E=\mathbb{C}[v_1,v_2,v_3]/\mathcal{I}$, i.e., E is the ring of regular functions on the affine elliptic curve in $\mathbb{C}^3$ defined by polynomials (32). The image of $v_j\in\mathbb{C}[v_1,v_2,v_3]$ in E is denoted by $\bar v_j$. Consider also a basis x, y, z of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})\cong\mathfrak{so}_3(\mathbb{C})$ with the relations

$$[x,y]=z, \quad [y,z]=x, \quad [z,x]=y$$

and endow the space $L=\mathfrak{sl}_2\otimes_{\mathbb{C}}E$ with the natural Lie algebra structure

$$[a\otimes f,\, b\otimes g]=[a,b]\otimes fg,\quad a,\, b\in\mathfrak{sl}_2(\mathbb{C}),\ f,\, g\in E.$$

PROPOSITION 1 ([**9**]). *Consider the Lie algebra P over $\mathbb{C}$ given by generators P_1, P_2, P_3 and relations* (30)*. Suppose that the numbers j_1, j_2, j_3 are distinct. Then the mapping*

$$P_1\mapsto x\otimes\bar v_1,\quad P_2\mapsto y\otimes\bar v_2,\quad P_3\mapsto z\otimes\bar v_3. \tag{33}$$

gives an isomorphism of P onto the subalgebra $R\subset L$ generated by the elements $x\otimes\bar v_1$, $y\otimes\bar v_2$, $z\otimes\bar v_3\in L$.

5. The prolongation algebra of the nonsingular KN equation

It is shown in [**4**] and rediscovered in [**7**] that Sklyanin's zero-curvature representation

$$D_tM-D_xN+[M,N]=0$$

for (26) (see [**10, 4, 9**] for the precise form of M, N) leads to a zero-curvature representation for the equation

$$u_t=u_3-\frac{3}{2}\frac{u_2^2}{u_1}+\frac{bu^4-cu^2+b}{u_1}. \tag{34}$$

Namely, denote by $\tilde M=\tilde M(u,u_1)$ the matrix function obtained from $M=M(S_1,S_2,S_3)$ by the substitution

$$S_1=\frac{u}{u_1},\quad S_2=i\frac{u^2+1}{2u_1},\quad S_3=\frac{u^2-1}{2u_1}. \tag{35}$$

Then there is a matrix-function $N'(u,u_1,u_2,u_3)$ such that the pair $\tilde M$, N' forms a zero-curvature representation for (34). Here and below $i=\sqrt{-1}\in\mathbb{C}$.

REMARK 5. Transformation (35) does not map solutions of (34) to solutions of the LL equation, since, for instance,

$$\Big(\frac{u}{u_1}\Big)^2+\Big(i\frac{u^2+1}{2u_1}\Big)^2+\Big(\frac{u^2-1}{2u_1}\Big)^2=0, \tag{36}$$

while in (26) we have $S_1^2+S_2^2+S_3^2=1$. However, below we use (35) to establish one-to-one correspondence between prolongation structures of the two equations.

On the other hand, formulas (35) and (36) inspire one to consider the analog of the LL equation (26) with the requirement $S_1^2+S_2^2+S_3^2=0$. Unfortunately, the prolongation algebra for the resulting system is trivial, and (35) still does not map solutions of (34) to solutions of this system. Perhaps a better understanding of relations between the LL equation and the KN equation can be derived from the results of [**4**].

Motivated by formulas (27) and (35), we proceed in describing the prolongation algebra of the KN equations as follows. Suppose first that (12) takes the form

$$u_t = u_3 - \frac{3}{2}\frac{u_2^2}{u_1} + \frac{bu^4 - cu^2 + b}{u_1} + au_1, \; a,\, b,\, c \in \mathbb{C}, \tag{37}$$

We rewrite (24) in the more convenient form

$$A_1 = uP_1 + i\frac{u^2+1}{2}P_2 + \frac{u^2-1}{2}P_3, \tag{38}$$

i.e., $A_{11} = P_1$, $A_{12} = (iP_2 + P_3)/2$, $A_{10} = (iP_2 - P_3)/2$.

Evidently, the elements

$$A_0,\; B_0,\; P_1,\; P_2,\; P_3 \tag{39}$$

represent another set of generators for the prolongation algebra. Let us write down the corresponding relations. From (22) one gets

$$[P_j, A_0] = [P_j, B_0] = [A_0, B_0] = 0, \quad j = 1, 2, 3, \tag{40}$$

while equation (23) gives the following relations between P_j

$$\begin{aligned}
&[P_1, [P_2, P_3]] = [P_2, [P_3, P_1]] = [P_3, [P_1, P_2]] = 0,\\
&\frac{8}{3}bP_1 = [P_3, [P_3, P_1]] - [P_2, [P_2, P_1]],\\
&\frac{1}{3}(4b + 2c)P_2 = -[P_1, [P_1, P_2]] + [P_3, [P_3, P_2]],\\
&\frac{1}{3}(4b - 2c)P_3 = [P_1, [P_1, P_3]] - [P_2, [P_2, P_3]].
\end{aligned} \tag{41}$$

We see that these relations coincide with relations (30) for

$$j_2 - j_3 = -\frac{8}{3}b, \quad j_3 - j_1 = \frac{1}{3}(4b + 2c), \quad j_1 - j_2 = \frac{1}{3}(4b - 2c). \tag{42}$$

This implies the following theorem.

Theorem 1. *For any covering of the LL equation* (26) *given by* (27), (28) *we obtain a covering of equation* (37) *with* (42) *as follows. Substituting* (35) *to* (27)*, one gets the corresponding x-part* (19)*, which in turn determines the t-part by formula* (20)*, where $B_0(w)$ is an arbitrary vector field commuting with the x-part, for example one may take $B_0 = 0$.*

And vice versa, given a covering of equation (37) *with x-part* (19)*, one obtains a covering of the form* (27), (28) *for the LL equation* (26) *satisfying* (42)*. Namely, the fields P_1, P_2, P_3 are determined through decomposition* (38)*, while P_4, P_5 are taken such that they meet* (29)*, for example $P_4 = P_5 = 0$.*

An example of this construction is given in Section 6.

It is easily seen that numbers (42) are nonzero if and only if the roots of the polynomial $bu^4 - cu^2 + b$ are distinct. In this case the Lie

algebra with generators P_1, P_2, P_3 and relations (41) is described by Proposition 1, which implies the following statement.

THEOREM 2. *The prolongation algebra of equation* (37) *with generators* (39) *and relations* (40), (41) *is isomorphic to the direct sum* $C \oplus R$, *where* $C = \langle A_0, B_0\rangle$ *is a commutative* 2*-dimensional algebra and* R *is the algebra defined in Proposition 1 by means of mapping* (33), *the numbers* $j_\alpha - j_\beta$ *being given by* (42).

Return to the nonsingular KN equation (1). Following [**7**], consider the linear-fractional transformation

$$u \mapsto e_1 + \frac{1}{4}(p^4 - q^4)\frac{q - up}{q + up}, \tag{43}$$

where $p,\, q \in \mathbb{C}$ are some solutions of the system

$$p^2q^2 = e_2 - e_3, \quad p^4 + q^4 = 6e_1 \tag{44}$$

and $e_1,\, e_2,\, e_3 \in \mathbb{C}$ are the roots of the polynomial $4u^3 - g_2u - g_3$.

LEMMA 1. *Transformation* (43) *is nontrivial if and only if the numbers* $e_1,\, e_2,\, e_3$ *are distinct, i.e., the KN equation* (1) *is nonsingular. In this case transformation* (43) *turns equation* (1) *into equation* (37) *with*

$$b = e_2 - e_3, \quad c = 6e_1. \tag{45}$$

PROOF. Clearly, transformation (43) is nontrivial if and only if

$$p \neq 0, \quad q \neq 0, \tag{46}$$

$$p^4 - q^4 \neq 0. \tag{47}$$

By definition (44), condition (46) is equivalent to $e_2 \neq e_3$. In addition, we have $(p^4 - q^4)^2 = 4(3e_1 + e_2 - e_3)(3e_1 - e_2 + e_3)$, which implies, taking into account $e_1 + e_2 + e_3 = 0$, that (47) is equivalent to $e_1 \neq e_2,\, e_1 \neq e_3$. This proves the first statement of the lemma, while the second statement is straightforward to check. □

REMARK 6. Existence of a linear-fractional transformations from (1) to (37) was claimed already in [**4**], but a formula was not given there. Note that the class of equations (12) and the form (19), (24) of the prolongation structure are preserved by arbitrary linear-fractional transformations $u \mapsto (k_1u + k_2)/(k_3u + k_4)$.

Since isomorphic equations have the same prolongation algebra, Lemma 1 implies that the prolongation algebra of the nonsingular KN equation (1) is isomorphic to the prolongation algebra of equation (37) with (45) described in Theorem 2 and Proposition 1. In this case polynomials (32) are

$$v_i^2 - v_j^2 + \frac{8}{3}(e_j - e_i), \quad i,\, j = 1, 2, 3. \tag{48}$$

THEOREM 3. *Let $\mathcal{I} \subset \mathbb{C}[v_1, v_2, v_3]$ be the ideal generated by polynomials* (48) *and $E = \mathbb{C}[v_1, v_2, v_3]/\mathcal{I}$. The prolongation algebra of the nonsingular KN equation* (1) *is isomorphic to the direct sum $C \oplus R$, where C is a commutative 2-dimensional algebra and R is the subalgebra of $\mathfrak{sl}_2 \otimes_{\mathbb{C}} E$ generated by $x \otimes \bar{v}_1$, $y \otimes \bar{v}_2$, $z \otimes \bar{v}_3$.*

The explicit isomorphism is derived through transformation (43) from the one for equation (37) described in Theorem 2.

REMARK 7. The known for (1) $\mathfrak{sl}_2$-valued ZCR with elliptic parameters [**4, 7**] arises from the restriction to R of the family of homomorphisms

$$\rho_a\colon \mathfrak{sl}_2 \otimes_{\mathbb{C}} E \longrightarrow \mathfrak{sl}_2, \quad x \otimes p \mapsto p(a)x,$$

parameterized by the points $a \in \mathbb{C}^3$ of the elliptic curve.

6. A polynomial zero-curvature representation

In particular, Theorem 1 establishes one-to-one correspondence between ZCRs for the LL equation (26) and equation (37) with (42). Consider the following example.

In [**3**] a zero-curvature representation $D_x M - D_t N + [M, N] = 0$ was found for the LL equation (26) with

$$M = \frac{1}{2} S(\lambda + \tilde{J}) \tag{49}$$

(the form of N is not important for us), where

$$S = \frac{1}{2}\begin{pmatrix} 0 & S_1 & S_2 & S_3 \\ -S_1 & 0 & S_3 & -S_2 \\ -S_2 & -S_3 & 0 & S_1 \\ -S_3 & S_2 & -S_1 & 0 \end{pmatrix},$$

$$\tilde{J} = \operatorname{diag}(-j_1' - j_2' + j_3',\ -j_1' + j_2' - j_3',\ j_1' - j_2' - j_3',\ j_1' + j_2' + j_3'),$$

$$j_k' = \sqrt{-4 j_k}, \quad k = 1, 2, 3,$$

and λ is an unconstrained complex parameter.

Denote by $\tilde{M} \in \mathfrak{sl}_4(\mathbb{C})$ the matrix obtained from (49) by substitution (35). By formulas (20), (10) and the procedure of Theorem 1, the matrices $\tilde{M}$ and

$$N' = -\frac{u_3}{u_1}\tilde{M} + \frac{u_2^2}{2u_1^2}\tilde{M} + 2u_2\frac{\partial \tilde{M}}{\partial u} - \frac{bu^4 - cu^2 + b}{3u_1^2}\tilde{M} - 2u_1[\tilde{M}, \frac{\partial \tilde{M}}{\partial u}] - 2u_1^2\frac{\partial^2 \tilde{M}}{\partial u^2} + a\tilde{M},$$

constitute a ZCR $D_t\tilde{M} - D_x N' + [\tilde{M}, N'] = 0$ for (37) with (42).

It is easily seen that for any b, $c \in \mathbb{C}$ there exist j_1, j_2, $j_3 \in \mathbb{C}$ such that (42) holds. Therefore, we have constructed a ZCR for each

equation (37). Applying transformation (43), we obtain a new ZCR for the nonsingular KN equation (1).

Interestingly, this ZCR is polynomial in the spectral parameter in contrast to the known for (1) ZCRs with elliptic parameters [**4, 6, 7**].

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